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Finite-size scaling in systems with long-range interaction

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Abstract

The finite-size critical properties of the $\mathcal{O}(n)$ vector φ^4 model, with long-range interaction decaying algebraically with the interparticle distance r like $r^{-d-\sigma}$, are investigated. The system is confined to a finite geometry subject to periodic boundary condition. Special attention is paid to the finite-size correction to the bulk susceptibility above the critical temperature T_c. We show that this correction has a power-law nature in the case of pure long-range interaction i.e. $0 < \sigma < 2$ and it turns out to be exponential in case of short-range interaction i.e. $\sigma = 2$. The results are valid for arbitrary dimension d, between the lower ($d_< = \sigma$) and the upper ($d_> = 2\sigma$) critical dimensions.

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I. INTRODUCTION

The critical behaviour at a second order phase transition depends upon the dimensionality d, degrees of freedom n, symmetry of the Hamiltonian (either in spin-space or in coordinate space) and interaction potentials. Generally speaking, the nature of the potential of the model under consideration can describe different physical situations (for a review see reference [1]). The simplest interaction potential, which has attracted the attention of investigators, is the one corresponding to long-range ferromagnetic interaction decaying algebraically with the spin interdistance r as $r^{-d-\sigma}$, where d is the dimension of the system and σ is the parameter controlling the range of the interaction. The interest in such type of interaction is tightly related to the exploration of the critical behaviour of systems with restricted dimensionality, in which no phase transitions occur otherwise.

The investigation of systems with long-range interaction was initiated by Joyce in his paper on the ferromagnetic spherical model [2]. The results of Joyce were generalized to the $\mathcal{O}(n)$ vector φ^4 model by means of perturbation theory in combination with the renormalization group techniques [3–7] and the $1/n$-expansion [8]. These investigations were also extended to dynamical critical phenomena (see Ref. [9] and references therein). Computer simulations also contributed in the exploration of the critical properties of such systems [10–12]. The results of these simulations, obtained by means of the Monte Carlo method, concerned mainly systems with classical critical behaviour, in the sense that the critical exponents are those belonging to the Landau theory. Rigorous results were obtained for low dimensional systems with long-range interaction (see reference [13,14] and references therein).

The analytic exploration of the scaling properties of confined systems with long-range interaction took its starting point on the spherical model. The reason for choosing that model is the relatively simple nature of the mathematical expression entering the equations characterizing its thermodynamics (for a complete set of references on the subject see reference [15]). Very recently these investigations were extended to $\mathcal{O}(n)$ vector models [16,17], using the renormalization group approach and the ε expansion to the one-loop order. In reference [16] the Binder cumulant has been evaluated at the vicinity of the critical temperature. It has been found that the expression for that quantity can be deduced just by choosing an appropriate rescaling of the parameters in that evaluated for short-range interaction case. The authors of reference [17] evaluated the susceptibility at the critical temperature T_c, as well as in the region, determined by the condition $L/\xi \gg 1$, where L is the linear size of the system and ξ - the bulk correlation length. In this region the bulk critical behaviour dominates the finite-size critical one. It has been shown that the finite-size correction to the bulk critical properties of the system has a power-law nature. This result is distinct from that obtained for the case of short-range interaction, where the finite-size correction falls-off exponentially.

In this paper we will investigate the finite-size scaling in a system with ferromagnetic long-range interaction at a fixed dimension d with $d_< < d < d_>$. The parameters $d_< = \sigma$ and $d_> = 2\sigma$ are, respectively, the lower and upper critical dimensions of the model, with $\sigma \leq 2$. To this end we will use the approach developed in references [18–20]. This method has been used successfully for the evaluation of critical exponents, as well as, critical amplitudes for various thermodynamic functions in the φ^4 model with short-range potential. The most

important property of this approach is that the quadratic (temperature-dependent) term does not enter explicitly the expansions, which can be used for both sides of the critical temperature T_c. This method has been also used for the investigation of the theory of finite size-scaling [21–23]. While a perfect agreement between the analytical results of reference [21] and those obtained by Monte Carlo simulations has been reported in reference [24], references [22,23] showed disagreements with some of the known results. In particular we would like to emphasize their finding of the non-exponential decay of finite-size corrections to the bulk critical behaviour.

We will consider spin system consisting of n-component unit vectors associated with d-dimensional lattice and interacting via a pair potential of the general translationally invariant form. The Hamiltonian of this model is given by

$$\beta\mathcal{H}\{\varphi\} = \frac{1}{2}\int_V d^d\boldsymbol{x}\left[a\left(\nabla\varphi\right)^2 + b\left(\nabla^{\sigma/2}\varphi\right)^2 + r_0\varphi^2 + \frac{1}{2}u_0\varphi^4\right], \tag{1.1}$$

where φ is a short hand notation for the space dependent n-component field $\varphi(\boldsymbol{x})$, $r_0 = r_{0c} + t_0$ ($t_0 \propto T - T_c$), a, b and u_0 are model constants. $V \equiv L^d$ is the volume of the system. In equation (1.1), we assumed $\hbar = k_B = 1$ and the size scale is measured in units in which the velocity of excitations $c = 1$. We note that the operator ∇^σ is defined by its Fourier transform

$$\nabla^\sigma f(\boldsymbol{x}) \equiv -L^{-d}\sum_{\boldsymbol{k}}\int d^d\boldsymbol{x}' e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}|\boldsymbol{k}|^\sigma f(\boldsymbol{x}').$$

The parameter β is set for the inverse temperature. Here we will consider periodic boundary conditions. This means

$$\varphi(x) = L^{-d}\sum_{\boldsymbol{k}}\varphi(\boldsymbol{k})\exp\left(i\boldsymbol{k}\cdot\boldsymbol{x}\right), \tag{1.2}$$

where $\boldsymbol{k}$ is a discrete vector with components $k_i = 2\pi n_i/L$ ($n_i = 0, \pm1, \pm2, \cdots,\ i = 1, \cdots, d$) and a cutoff $\Lambda \sim a^{-1}$ (a is the lattice spacing). In this paper, we are interested in the continuum limit *i.e.* $a \to 0$. As long as the system is finite we have to take into account the following assumptions $L/a \to \infty$, $\xi \to \infty$ while ξ/L is finite.

The critical behaviour of the model Hamiltonian (1.1), has been investigated in details in the early 70's. The main focus of interest has been turned to the evaluation of the critical exponents. Using renormalization group techniques, it has been shown that the critical behaviour of this model is dominated by the long-range interaction for $0 < \sigma < 2$ [3,5]. In this case the critical exponents are σ dependent, in particular the Fisher exponent $\eta_\sigma = 2-\sigma$. As long as σ becomes of the same order as $2 - \eta_2$, where η_2 is the Fisher exponent of the short-range model, a crossover from the long-range critical behaviour to the short-range critical one takes place [4,6,7]. For $\sigma \geq 2 - \eta_2$ the critical behaviour is dominated by the short-range interaction and the critical exponents are those of the pure short-range model. Another way to establish the relevance of the long range term for $\sigma < 2 - \eta_2$ is presented in reference [25].

The plan of this paper is as follows. In Section II we describe the renormalization scheme for the bulk φ^4 theory with long-range interaction. In Section III we discuss the effects of

confined geometries on the bulk critical behaviour. We investigate the influence of the long-range interaction on the finite-size correction to the bulk critical behaviour. In Section IV we discuss our results briefly. An appendix is added in order to complement the results of Section III.

II. THE BULK SYSTEM

A. Bare theory

For simplicity here we will consider the model with pure long-range ferromagnetic interaction i.e. the parameter σ controlling the range of the interaction is smaller than 2, value characterizing the short range interaction. In other words, we will consider the model (1.1) with the parameters $a = 0$ and $b = 1$. We believe that one has first to understand this model before starting to explore the model with both long and short-range interactions present. The investigation of the critical properties of the complex structure of this model will presented elsewhere.

It has been shown that the Hamiltonian with pure long-range interaction can be treated with field theoretical renormalization group techniques [5]. The renormalization constants as well as the field theoretic functions were calculated. A superficial discontinuity of the anomalous dimension of field theories occurs as soon as $\sigma = 2$. This however is true only at this particular point, and as long as we are far from that point, we can use the model with the spectrum $r_0 + |\boldsymbol{k}|^\sigma$. In this case the renormalization constants, to one loop order, are given by:

$$Z_\varphi = 1 + \mathcal{O}(u^2) \tag{2.1a}$$

$$Z_r = 1 + \frac{n+2}{\varepsilon} u + \mathcal{O}(u^2) \tag{2.1b}$$

$$Z_u = 1 + \frac{n+8}{\varepsilon} u + \mathcal{O}(u^2). \tag{2.1c}$$

Here, as usual, Z_φ is the scaling field amplitude, Z_u the coupling constant renormalization, and Z_r the renormalization of the φ^2 insertions in the critical theory. The parameter $\varepsilon = 2\sigma - d$ denotes the deviation from the upper critical dimension.

The application of the method proposed by the authors of references [18–20] to systems with pure long-range interaction can be established easily following the way this has been done in combination with the ε expansion [5]. In particular we turn our attention to the inverse bare susceptibility, related to the two-point vertex function, at finite external wave-number $\boldsymbol{k}$

$$\frac{1}{\overset{0}{\chi}(\boldsymbol{k})} = \overset{0}{\Gamma}{}^{(2)}(\boldsymbol{k}, r_0, u_0, \Lambda, d). \tag{2.2}$$

This function is, of course, well defined at $T \gtrsim T_c$ for dimensions $d > \sigma$. The parameter r_0 is, as usual, a linear function of the reduced temperature $t = (T - T_c)/T_c$. Its critical

value is determined by the condition $\overset{0}{\chi}(\mathbf{0})^{-1} = 0$, from which we obtain a natural implicit definition for the function

$$r_{0c} \equiv r_{0c}(u_0, \Lambda, d). \tag{2.3}$$

Following reference [18], we will consider the difference $r_0 - r_{0c}$, which is a function of the correlation length (i.e. $r_0 - r_{0c} = h(\xi, u_0, d)$), instead of r_0 itself into the expressions for the different vertex functions. So, we will consider the vertex functions $\overset{0}{\Gamma}{}^{(N)}$ as depending on the parameters $r_0 - r_{0c}$, u_0, Λ and d. We will denote the dimensionally regularized vertex function by $\overset{0}{\Gamma}{}^{(N)} \equiv \overset{0}{\Gamma}{}^{(N)}(r_0 - r_{0c}, u_0, d)$. Correspondingly the dimensionally regularized critical parameter r_{0c} will be denoted by $r_{0c}(u_0, d)$. From simple dimensional arguments, one obtains the relation

$$r_{0c} \propto u_0^{\sigma/\varepsilon}. \tag{2.4}$$

The method used here differs from the ones when the ε expansion come into play, by the fact that the critical parameter r_{0c} is non-vanishing within the dimensional regularized theory. It is clear that the use of the ε expansion implies an apparent vanishing of r_{0c}, as seen formally from the relation (2.4) between r_{0c} and u_0, which for infinitesimal ε does not yield a contribution at finite order to the perturbation theory.

In order to make clear the definitions introduced along this section, we will present here the one-loop results. For the two point vertex function we get

$$\overset{0}{\Gamma}{}^{(2)} = r_0 + \boldsymbol{k}^\sigma - (n+2)u_0 A_{d,\sigma} \frac{r_0^{-\varepsilon/\sigma}}{\varepsilon} + \mathcal{O}(u_0^2), \tag{2.5}$$

where the geometrical factor $A_{d,\sigma}$ is defined by

$$A_{d,\sigma} = \frac{2}{(4\pi)^{d/2}\Gamma(d/2)} \Gamma\left(1 + \frac{\varepsilon}{\sigma}\right) \Gamma\left(1 - \frac{\varepsilon}{\sigma}\right). \tag{2.6}$$

The four point vertex function is

$$\overset{0}{\Gamma}{}^{(4)} = u_0 - (n+8)u_0^2 A_{d,\sigma} \left(1 - \frac{\varepsilon}{\sigma}\right) \frac{r_0^{-\varepsilon/\sigma}}{\varepsilon} + \mathcal{O}(u_0^2). \tag{2.7}$$

In the next subsection we will discuss the renormalization of the bare theory for fixed dimension d confined between the lower and the upper critical dimensions given by σ and 2σ, respectively.

B. Renormalized theory

It is well known, that the perturbative results of the bare theory do not provide a correct description in the critical region $\xi \to \infty$, for dimensions below the upper critical dimension. This can be outwitted by taking advantage of the ideas of the renormalized theory, which furnishes a mapping from the critical region to non-critical one. Our starting point is the

expression for the N-point vertex functions $\tilde{\Gamma}_0^N(\xi, u_0, d)$ obtained from $\overset{0}{\Gamma}{}^{(N)}(r_0 - r_{0c}, u_0, d)$ by switching from the variable $r_0 - r_{0c}$ to ξ. This is possible because of the fact that the reduced temperature is tightly related to the correlation length. The deviation of the parameter r_0 from its critical value will be a fixed quantity and hence the correlation length will have the same property.

We treat the theory by using the minimal subtraction scheme at fixed dimension d. To this end, we introduce the renormalized quantities

$$\varphi(x) = Z_\varphi^{1/2}\varphi_R(x) \tag{2.8a}$$

$$r_0 = r_{0c} + Z_r r, \quad r_0 - r_{0c} > 0 \tag{2.8b}$$

$$A_{d,\sigma}u_0 = \mu^\varepsilon Z_u Z_\varphi^{-2} u. \tag{2.8c}$$

Then the renormalized vertex functions takes the form

$$\tilde{\Gamma}^{(N)}(\xi, u, \mu, d) = Z_\varphi^{N/2}\tilde{\Gamma}_0^N(\xi, \mu^\varepsilon Z_u Z_\varphi^{-2} u A_{d,\sigma}^{-1}, d). \tag{2.9}$$

In addition, we require that the renormalization constants Z_φ and Z_u absorb just the poles of $\tilde{\Gamma}_0^N$ at the upper critical dimension, which turns out to be 2σ for the model under consideration. In equations (2.8) and (2.9) the parameter μ is an inverse reference length, which will be chosen as the amplitude of the asymptotic bulk correlation length.

In the following we proceed in a standard way, by deriving a differential renormalization-group equation for the vertex function $\tilde{\Gamma}^{(N)}$. This is achieved by taking the derivative of equation (2.9) with respect to μ at fixed constant u_0 and $r_0 - r_{0c}$. This leads to

$$[\mu\partial\mu + \beta_u(u, \varepsilon)\partial_u + \frac{1}{2}N\zeta_\varphi(u)]\tilde{\Gamma}^{(N)}(\xi, u, \mu, d) = 0, \tag{2.10}$$

with

$$\beta_u(u, \varepsilon) = (\mu\partial_u u)_0, \qquad \zeta_\varphi(u) = (\mu\partial_\mu \ln Z_\varphi^{-1})_0. \tag{2.11}$$

Here the subscript 0 indicates that the differentiation is performed at fixed parameters of the bare theory. Using the method of characteristics, a formal solution of the renormalization-group differential equation (2.10) is given by

$$\tilde{\Gamma}^{(N)}(\xi, u, \mu, d) = \tilde{\Gamma}^{(N)}(\xi, u(\ell), \ell\mu, d)\exp\left(\frac{N}{2}\int_1^\ell \zeta_\varphi(\ell')\frac{d\ell'}{\ell'}\right). \tag{2.12}$$

Here $\zeta(\ell)(u(\ell), d)$ and $u(\ell)$ is the solution of flow equation

$$\ell\frac{du(\ell)}{d\ell} = \beta_u[u(\ell), d] \tag{2.13}$$

with the initial condition $u(\ell) = u$. The most convenient choice for the flow parameter ℓ is

$$\ell\mu = \xi^{-1}. \tag{2.14}$$

The renormalized vertex functions can be written as

$$\tilde{\Gamma}^{(N)}(\xi, u, \mu, d) = \xi^{-d+(d-\sigma)\frac{N}{2}} f^{(N)}(\mu\xi, u, d), \tag{2.15}$$

where the amplitude functions $f^{(N)}$ are dimensionless. The renormalizability of the φ^4 model with long-range interaction for dimensions d less than the upper critical dimension 2σ is a warranty for the finiteness of the $\tilde{\Gamma}^{(N)}$ at fixed u, μ and ξ. From (2.12), (2.14) and (2.15), we obtain the expression

$$f^{(N)}(\mu\xi, u, d) = f^{(N)}(1, \mu(\ell), d) \exp\left(\frac{N}{2}\int_1^\ell \zeta_\varphi(\ell')\frac{d\ell'}{\ell'}\right). \tag{2.16}$$

As usual, equation (2.16) lies in the basis of the mapping of the amplitude function $f^{(N)}(\mu\xi, u, d)$ from the critical region, where the perturbation theory breaks down to the noncritical region, where the perturbation theory is applicable.

C. Asymptotic regime

In the asymptotic limit, determined by ($\ell \to 0$, $\xi \to \infty$), the coupling $u(\ell)$ approaches the fixed point $u^* = u(0)$, which is the zero of the β function of the theory i.e.

$$\beta_u(u^*, 0, d) = 0. \tag{2.17}$$

For $\xi \to \infty$, equation (2.16), takes the asymptotic form

$$f^{(N)}(\mu\xi, u, d) \sim A^{(N)} f^{(N)}(1, u^*, 0, d)(\mu\xi)^{N\eta/2}, \tag{2.18}$$

with the critical exponent $\eta = -\zeta_\varphi(u^*, 0, d)$ and the nonuniversal amplitude

$$A^{(N)} = \exp\left(\frac{N}{2}\int_1^0 [\zeta_\varphi(\ell') - \zeta_\varphi(0)]\frac{d\ell'}{\ell'}\right). \tag{2.19}$$

The results obtained here can be applied very easily to the particular case of the bare susceptibility corresponding to the particular value $N = 2$. This is given by

$$\chi = Z_\varphi(u, d)\xi^2 [f^{(2)}(1, u(\ell), d)]^{-1} \exp\left(\int_\ell^0 \zeta_\varphi(\ell')\frac{d\ell'}{\ell'}\right). \tag{2.20}$$

The critical behaviour above T_c reads

$$\chi = \mathcal{A}_\chi^+ t^{-\gamma}, \tag{2.21}$$

where

$$\mathcal{A}_\chi^+ = \xi_0^2 Z_\varphi(u, d)[A^{(2)} f^{(2)}(1, u^*, 0, d)]^{-1}. \tag{2.22}$$

Here we have used $\mu = \xi_0^{-1}$ and the asymptotic form $\xi = \xi_0 t^{-\nu}$ with $\gamma = \nu(2 - \eta)$.

III. THE FINITE SYSTEM

A. The effective Hamiltonian

For the evaluation of the effective Hamiltonian of the finite system, we need to calculate the free energy per unit volume. This is defined by

$$f = -L^{-d} \ln \mathcal{Z}; \qquad \mathcal{Z} = \int \mathcal{D}\varphi \exp(-\beta\mathcal{H}). \tag{3.1}$$

Here we are also interested in the expression of the susceptibility, which is defined by

$$\chi = \int d^d\boldsymbol{x} \, \langle \varphi(\boldsymbol{x})\varphi(0) \rangle = \frac{1}{\mathcal{Z}} \int d^d\boldsymbol{x} \int \mathcal{D}\varphi \varphi(\boldsymbol{x})\varphi(0) \exp(-\beta\mathcal{H}). \tag{3.2}$$

Following references [26,27], we split the field $\varphi = \Phi + \Sigma$ into a mode independent part

$$\Phi = L^{-d} \int d^d\boldsymbol{x}\varphi(\boldsymbol{x}), \tag{3.3}$$

which is equivalent to the magnetization and a part depending upon the non zero modes

$$\Sigma = L^{-d} {\sum_{\boldsymbol{k}}}' \varphi(\boldsymbol{k}) \exp(i\boldsymbol{k}\cdot\boldsymbol{x}). \tag{3.4}$$

Consequently (1.1) is decomposed as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I(\Phi, \Sigma), \tag{3.5}$$

with the zero-mode Hamiltonian

$$\mathcal{H}_0 = \frac{1}{2} L^d \left(r_0 \Phi^2 + \frac{1}{2} u_0 \Phi^4 \right). \tag{3.6}$$

Whence the partition function takes the form

$$\mathcal{Z} = \int_{-\infty}^{\infty} d\Phi \exp[-(\mathcal{H}_0 + \overset{0}{\Gamma}(\Phi^2))], \tag{3.7}$$

where

$$\overset{0}{\Gamma}(\Phi^2) = -\ln \int \mathcal{D}\Sigma \exp[-\mathcal{H}_I] \tag{3.8}$$

contains the contribution from higher order.

Now instead of the decomposition of reference [27] we will use the modified perturbation theory proposed in reference [21]. There, an appropriate decomposition of the $\mathcal{O}(n)$ vector φ^4 model with short-range interaction has been presented. This way has been proven to give a good quantitative results above, as well as below the critical temperature T_c. Applying

that method to our model we find, that the higher-mode dependent part $\mathcal{H}_I$ can be split into

$$\mathcal{H}_I = \frac{1}{2}\int d^dx[(r_0 + (n+2)u_0M_0^2)\Sigma^2 + (\nabla^{\sigma/2}\Sigma)^2] \\ +\frac{1}{2}\int d^dx[3u_0(\Phi^2 - M_0^2)\Sigma^2 + 2u_0\Phi\Sigma^3 + \frac{1}{2}u_0\Sigma^4], \tag{3.9}$$

where we have introduced the magnetization

$$M_0^2 = \left\langle \phi^2 \right\rangle. \tag{3.10}$$

Further a diagrammatic expansion of $\overset{0}{\Gamma}(\Phi^2)$ can be represented by two point vertices proportional to $u_0(\Phi^2 - M_0^2)$ in addition to the three and four point vertices $\sim\ u_0\Phi$ and $\sim\ u_0$. In this way, the finite-size perturbation theory is obtained as an expansion of $\overset{0}{\Gamma}(\Phi^2)$ in powers of $u_0(\Phi^2 - M_0^2)$, u_0 and $u_0^2\Phi^2$. In the following, instead of $\overset{0}{\Gamma}(\Phi^2)$ we consider $L^{-d}\overset{0}{\Gamma}(\Phi^2)$ which remains finite in the limit $L \to \infty$. To the leading order the finite-size correction reads

$$L^{-d}\overset{0}{\Gamma}(\Phi^2) = \frac{1}{2}{\sum_{\boldsymbol{k}}}' \ln(r_0 + (n+2)u_0M_0^2 + \boldsymbol{k}^\sigma) + \frac{n+2}{2}u_0\Phi^2 S_1(r_0) \\ -\frac{n+8}{4}u_0^2(\Phi^4 + 2M_0^2\Phi^2)S_2(r_0) + \mathcal{O}(u_0, u_0^2\phi^2, u_0^3(\Phi^2 - M_0^2)^3), \tag{3.11}$$

where

$$S_m(r) = L^{-d}{\sum_{\boldsymbol{k}}}'(r + (n+2)u_0M_0^2 + \boldsymbol{k}^\sigma)^{-m}. \tag{3.12}$$

Here we will investigate only the high temperature regime i.e. $T \gtrsim T_c$, so we will set $M_0^2 = 0$. A rearrangement of equations (3.6) and (3.11) leads to the effective Hamiltonian

$$\mathcal{H}_{\text{eff}}(r_0, u_0, L, \Phi^2) = \frac{1}{2}L^d[R\Phi^2 + \frac{1}{2}U\Phi^4 + \mathcal{O}(\Phi^6)], \tag{3.13}$$

where

$$R = r_0 + (n+2)u_0L^{-d}{\sum_{\boldsymbol{k}}}'(r_0 + \boldsymbol{k}^\sigma)^{-1}, \tag{3.14a}$$

$$U = u_0 - (n+8)u_0^2L^{-d}{\sum_{\boldsymbol{k}}}'(r_0 + \boldsymbol{k}^\sigma)^{-2}. \tag{3.14b}$$

Finally, the free energy (3.1) reads

$$f = L^{-d}\overset{0}{\Gamma}(0) - L^{-d}\ln \mathcal{Z}_{\text{eff}}, \tag{3.15a}$$

$$\mathcal{Z}_{\text{eff}} = \int_{-\infty}^{\infty} d\Phi \exp[\mathcal{H}_{\text{eff}}(r_0, u_0, L, \Phi^2)] \tag{3.15b}$$

and averages such as $\langle \Phi^{2p} \rangle$ are calculated from (3.2) as

$$\left\langle \Phi^{2p} \right\rangle = \frac{1}{\mathcal{Z}_{\text{eff}}} \int_{-\infty}^{\infty} d\Phi \Phi^{2p} \exp(-\mathcal{H}_{\text{eff}}(r_0, u_0, L, \Phi^2)). \tag{3.16}$$

Let us note that for the purpose of comparison with numerical results it is convenient to keep the exponential structure of the integrand in the various thermodynamic functions. Notice that an appropriate rescaling of the field Φ in equation (3.16), permits to write the momenta $\langle \Phi^{2p} \rangle$ as a function the 'scaling variable'

$$z = RL^{d/2} U^{-1/2}. \tag{3.17}$$

Whence

$$\left\langle \Phi^{2p} \right\rangle = (u_0 L^d)^{-p/2} \frac{\int_0^\infty dx x^{n+2p-1} \exp\left(-\frac{1}{2} z x^2 - \frac{1}{4} x^4\right)}{\int_0^\infty dx x^{n-1} \exp\left(-\frac{1}{2} z x^2 - \frac{1}{4} x^4\right)}. \tag{3.18}$$

These functions can be expressed in terms of the confluent hypergeometric function (see Appendix A).

B. Renormalization

As it has been mentioned in the previous section, in this paper, we will use the approach of references [18–20]. Here we will consider the limit of an infinite cutoff i.e. $\Lambda \to \infty$ at fixed $r_0 - r_{0c}$, where r_{0c} is the bulk critical value of r_0. Let us recall that the advantage of using this method is its direct application to fixed space dimensionality d, without resorting to the ε expansion. For systems with short-range interaction this method has been tested and accurate results, for various thermodynamic functions has been, obtained [21]. Since r_{0c} is of order $\mathcal{O}(u_0^{\sigma/2})$ in dimensionally regularized form, we find it safe to replace in our expressions r_0 by the difference $r_0 - r_{0c}$. The effective Hamiltonian will be function of $r_0 - r_{0c}$ but not of r_0 itself. The effective constants R and U defined in (3.14) are also functions of this deviation, which itself is a function of the correlation length. The perturbation approach consists of an expansion of $\mathcal{H}_{\text{eff}}$ with respect to the renormalized counterpart u of the coupling constant u_0 at fixed dimension. Since the finite nature of the geometry of the system does not alter the ultraviolet divergences, the usual bulk renormalizations are enough to describe the scaling properties the finite system.

Here we are interested in the evaluation of the explicit expressions of various thermodynamic quantities for the finite system. Different crossover regions, in terms of renormalized parameters, are under consideration. The corresponding renormalized effective coupling constants to equations (3.14) take the form

$$R = r(\ell) + (n+2)u(\ell)(\mu\ell)^{\sigma}\left[\frac{r(\ell)(\mu\ell)^{-\sigma}}{\varepsilon}\left[1-\left(\frac{r(\ell)}{(\mu\ell)^{\sigma}}\right)^{-\varepsilon/\sigma}\right]\right.$$
$$\left.+\frac{(\mu L)^{\varepsilon-\sigma}}{A_{d,\sigma}}F_{d,\sigma}(r(\ell)L^{\sigma})\right], \tag{3.19a}$$
$$U = u(\ell) + u^2(\ell)(n+8)\left[\frac{1}{\varepsilon}\left[1-\left(\frac{r(\ell)}{(\mu\ell)^{\sigma}}\right)^{-\varepsilon/\sigma}\right]+\frac{1}{\sigma}\left(\frac{r(\ell)}{(\mu\ell)^{\sigma}}\right)^{-\varepsilon/\sigma}\right.$$
$$\left.+\frac{(\mu L)^{\varepsilon}}{A_{d,\sigma}}F'_{d,\sigma}(r(\ell)L^{\sigma})\right], \tag{3.19b}$$

where we have used the definitions [17]

$$S_1(r) = -A_{d,\sigma}\varepsilon^{-1}r^{1-\varepsilon/\sigma} + L^{\sigma-d}F_{d,\sigma}(rL^{\sigma}). \tag{3.20a}$$
$$S_2(r) = A_{d,\sigma}\varepsilon^{-1}\left(1-\frac{\varepsilon}{\sigma}\right)r^{-\varepsilon/\sigma} - L^{\sigma-d}\frac{\partial}{\partial r}F_{d,\sigma}(rL^{\sigma}), \tag{3.20b}$$

and

$$F_{d,\sigma}(y) = \frac{1}{(2\pi)^{\sigma}}\int_0^{\infty} dx x^{\frac{\sigma}{2}-1}E_{\frac{\sigma}{2},\frac{\sigma}{2}}\left(-\frac{yx^{\sigma/2}}{(2\pi)^{\sigma}}\right)\left[\left(\sum_{\ell=-\infty}^{\infty}e^{-x\ell^2}\right)^d - 1 - \left(\frac{\pi}{x}\right)^{d/2}\right]. \tag{3.21}$$

Here the function

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \tag{3.22}$$

is the so called Mittag-Leffler type functions. For a more recent review on these functions and others related to them, and their application in statistical and continuum mechanics see reference [28]. A brief summary of some of the properties of the Mittag-Leffler functions are presented in reference [17].

By making the choice such that the flow parameter $\ell = \ell(t, L)$ satisfies the following relation

$$\xi^{-\sigma} + L^{-\sigma} = \mu^{\sigma}\ell^{\sigma}, \tag{3.23}$$

in the critical region, equations (3.19) imply the scaling forms

$$R(\ell) = \mu^{\sigma}\ell^{\sigma}\tilde{R}(tL^{1/\nu}), \qquad U(\ell) = \tilde{U}(tL^{1/\nu}). \tag{3.24}$$

Using the asymptotic expressions for $R(\ell)$ and $U(\ell)$, we obtain the asymptotic form of the variable z, defined in (3.17). It is given by

$$z(tL^{1/\nu}) = R(\ell)\mu^{-2}\ell^{-2}(L\mu\ell)^{d/2}A_d^{1/2}[U(\ell)]^{-1/2}. \tag{3.25}$$

From this expression, one can convince himself, easily, that the function $z(tL^{1/\nu})$ has an expansion in terms of $\sqrt{u^*}$.

C. The Susceptibility

In this section, we present our result for the finite-size scaling function of the magnetic susceptibility. Here we give the general expression, obtained by expanding with respect to the coupling constant u about its fixed point u^*. The dimension is kept fixed i.e. we are not going to expand in the vicinity of the upper critical dimension as it has been done earlier [17]. Furthermore, we won't expand the exponential weight of the integrand in equation (3.16). In this way consistency with the correct one-loop bulk expressions is ensured in the bulk limit, reached by sending the size L of the system to infinity. As it has been mentioned in the previous sections the arbitrary reference length μ^{-1} of the renormalized theory will be chosen as the amplitude of the correlation length i.e. $\mu^{-1} = \xi_0$.

In reference [29], it has been shown that, for periodic boundary conditions, the ultraviolet divergences of the susceptibility χ of finite systems in the continuum limit i.e. $a \to 0$ are identical to those of the bulk susceptibility χ_b. In reference [17], the application of this method has been extended to systems with long-range interaction and some results for the susceptibility in the asymptotic regimes $L/\xi \gg 1$, as well as $L/\xi \ll 1$ has been obtained. The results obtained there were limited to dimensions close to the upper critical one. Here we will extend these results to the whole interval of dimensions between the lower critical dimension $d_< = \sigma$ and the upper critical one $d_> = 2\sigma$.

The average $\langle \phi^2 \rangle$, entering the definition of the susceptibility (3.2), is defined with the statistical weight $\exp[-(\mathcal{H}_0 + \overset{0}{\Gamma})]$ given in equation (3.7). Obviously, in the bulk limit we recover the bulk susceptibility i.e. $\lim_{L\to\infty} \chi = \chi_b$.

For systems confined to a finite geometry, the renormalized susceptibility χ_R, as a function of $r_0 - r_{0c}$, u_0, and L, can be introduced as

$$\chi_R(\xi, u, L, \mu, d) = Z_\varphi^{-1} \chi(\xi, \mu^\varepsilon Z_u Z_\varphi A_{d,\sigma} u, L, d), \tag{3.26}$$

where Z_φ and Z_u are the bulk Z amplitude factors defined in Section II.

A renormalization-group equation for χ_R is obtained by deriving expression (3.26) for the susceptibility with respect to the parameter μ at fixed $r_0 - r_{0c}$ (function of the correlation length), u_0 and d. Since the linear size L of the system does not renormalize, we get

$$\left[\mu\partial_\mu + \beta_u \partial_u - \zeta_\varphi\right] \chi_R(\xi, u, L, \mu, d) = 0, \tag{3.27}$$

where the renormalization group functions $\beta(u)$ and $\zeta(u)$ are defined in equation (2.11) for the bulk theory. A formal solution for this equation is given by

$$\chi_R(\xi, u, L, \mu, d) = \chi_R(\xi, u(\ell), L, \ell\mu, d) \exp\left(\int_\ell^1 \zeta_\varphi(\ell') \frac{d\ell'}{\ell'}\right), \tag{3.28}$$

where the parameter ℓ can be chosen arbitrarily. The most convenient choice is that, for which ℓ satisfies the relation (3.23). Remark that the renormalization group equations (2.10) for the bulk system and (3.27) for the system confined to the finite geometry are similar. However because of the finiteness of the size L of the system a careful consideration of equations (3.27) and its solution (3.28) is in order. In this case, we introduce the dimensionless amplitude function $f_\chi(z)$ according to

$$\chi_R(L, \xi, u, \mu, d) = L^2 f_\chi(\mu L, \mu\xi, u, d). \tag{3.29}$$

In the asymptotic regime, given by $\ell \ll 1$, we obtain from the formal solution (3.28) of the renormalization group equation

$$\chi_R(L, \xi, u, \mu, d) \sim L^2 \ell^\eta \left[A^{(-2)}\right] f_\chi(\mu\ell L, \mu\ell\xi, u^*, d). \tag{3.30}$$

Using the fact that $\mu\ell L$ and $\mu\ell\xi$ are functions only of the ratio L/ξ, which follows from equation (3.23), we can write the bare susceptibility $\chi = Z_\varphi \chi_R$, as well as, the renormalized susceptibility χ_R in the following scaling form

$$\chi_R(L, \xi, u, \mu, d) \sim L^{2-\eta}\mu^\eta \left[1 + \left(\frac{L}{\xi}\right)^\sigma\right]^{\eta/\sigma} \left[A^{(-2)}\right] Y\left(\frac{L}{\xi}\right), \tag{3.31}$$

where $Y(z)$ is a scaling function of its argument.

Equation (3.31) is the final expression for the susceptibility of the finite system. This is the complete expression in whole $L^{-1} - \xi^{-1}$ plane. In the following we will consider different regimes with respect to the ratio ξ/L. For this purpose we investigate the $\mathcal{O}(n)$ vector φ^4 model. Here we turn a special attention to the limiting case $\xi/L \gg 1$.

In the remainder of this section we will investigate the behaviour of the susceptibility using an ordinary perturbation theory. Accordingly, the standard one-loop expression for the inverse susceptibility above the critical point for a system confined to a finite geometry reads

$$\chi^{-1} = r_0 + (n+2)u_0 L^{-d} \sum_{\boldsymbol{k}} (r_0 + \boldsymbol{k}^\sigma)^{-1} + \mathcal{O}(u_0^2). \tag{3.32}$$

The susceptibility for the bulk system is obtained by sending the size of the system to infinity i.e. the sum in the last equation is replaced by integrals. It is

$$\chi_b^{-1} = r_0 + (n+2)u_0 \int d^d\boldsymbol{k} (r_0 + \boldsymbol{k}^\sigma)^{-1} + \mathcal{O}(u_0^2). \tag{3.33}$$

After some straightforward calculation one gets

$$\chi(t, u_0, L, d)^{-1} = t\left[1 + (n+2)u_0 L^{2\sigma-d}(tL^\sigma)^{-2}\left(1 + tL^\sigma F_{d,\sigma}(tL^\sigma)\right)\right]. \tag{3.34}$$

From this expression, we see that it is not allowed to set $y = 0$. In other words to take the limit $t \to 0$, while the size of the system is kept fixed. In the opposite limit $L/\xi \gg 1$, the right hand side of equation (3.34) is well defined and the result gives the finite-size correction to the expression of the bulk susceptibility. However, the behaviour of the function $F_{d,\sigma}(y)$ is strongly dependent upon the value of the parameter σ. Indeed, it depends upon the nature of the interparticle interaction in the system. Here we will discuss the influence of the interaction on the behaviour of the finite-size correction.

The function $F_{d,\sigma}(y)$ has the following large y asymptotic behaviour [17]

$$F_{d,\sigma}(y) \simeq -\frac{1}{y} + \frac{2^\sigma \pi^{-d/2}\Gamma\left(\frac{d+\sigma}{2}\right)}{y^2\Gamma(-\frac{\sigma}{2})} {\sum_{\boldsymbol{l}}}' \frac{1}{|\boldsymbol{l}|^{d+\sigma}} \tag{3.35}$$

for the case $0 < \sigma < 2$, and

$$F_{d,2}(y) \simeq -\frac{1}{y} + d(2\pi)^{(1-d)/2} y^{(d-3)/4} e^{-\sqrt{y}} \tag{3.36}$$

for the particular case $\sigma = 2$. These results show that the last term in equation (A9) is just cancelled by the first term in equations (3.35) and (3.36).

In the case of long-range interaction $0 < \sigma < 2$, we obtain for the susceptibility, after renormalizing the theory,

$$\chi = \chi_b \left[1 - u^*(n+2) 2^\sigma \pi^{-d/2} \left(tL^\sigma\right)^{-1-\frac{d}{\sigma}} \frac{\Gamma\left(\frac{d+\sigma}{2}\right)}{\Gamma(-\frac{\sigma}{2})} \sum_{\boldsymbol{l}}{}' \, \boldsymbol{l}^{-d-\sigma} + \mathcal{O}(u^{*2}) \right] \tag{3.37}$$

in agreement with the finite-size scaling hypothesis. Equation (3.37) shows that the finite-size scaling behaviour of the system is dominated by the bulk critical behaviour, with small correction in powers of L. First the power law fall-off of the finite-size corrections to the bulk critical behaviour, due to long-range nature of the interaction, was found in the framework of the spherical model [30,31], which is believed to belong to the same class of universality as the $\mathcal{O}(n)$ vector model with $n \to \infty$.

It should be noted that the above result (3.37) cannot be continued smoothly to the case of short-range interaction $\sigma = 2$. In this particular case$F_{d,2}(y)$ (see equation (3.36)) falls off exponentially fast and, correspondingly, the finite-size corrections to χ are exponentially small:

$$\chi = \chi_b \left[1 - u^*(n+2) d (2\pi t L^2)^{(1-d)/4} e^{-L\sqrt{t}} + \mathcal{O}\left(u^{*2}\right) \right]. \tag{3.38}$$

This result was obtained first in reference [23]. Notice that by expanding to the first order in $\varepsilon = 2\sigma - d$ in equation (3.37) and (3.38), we recover, the corresponding results (3.10) and (3.31) of reference [17].

Here an important remark is in order. In appendix A, we show that the results obtained for the susceptibility in this section are identical to those obtained, using the mode expansion. This is in disagreement with the conclusions of reference [23] claiming that the mode expansion is inadequate for the description of the finite-size scaling in the region of the phase diagram determined by the condition $L/\xi \gg 1$. Especially for the case of short-range interaction this result is in agreement with that obtained by means of Monte Carlo.

IV. DISCUSSION

In this paper, we have investigated the finite-size scaling properties in the $\mathcal{O}(n)$-symmetric φ^4 model with long-range interaction potential decaying algebraically with the interparticle distance. By means of the method, which consists of the use of the minimal subtraction scheme applied to a fixed space dimensionality developed in references [18–20] for the bulk system and in [21–23] for the one with finite linear size, we generalized the results of reference [17] obtained for the susceptibility by means of the $\varepsilon = 2\sigma - d$ expansion. The obtained results are in a full agreement with those obtained in reference [17].

Here we restricted our calculation to the critical domain $T \gtrsim T_c$ and investigated the behaviour of the susceptibility to the one-loop order in the coupling constant u_0 of the bare theory. We have turned our attention to the special case, in the phase diagram, where the bulk critical behaviour is dominating the finite-size scaling one. We have found that the finite-size correction falls off algebraically (3.37) in the case when $\sigma < 2$ and exponentially (3.38) in the particular case $\sigma = 2$, characterizing the short-range interaction. These results are obtained by using two different methods: (i) The standard mode expansion of references [26,27] and (ii) The ordinary perturbation theory used in references [22,23]. We would like to mention here that our results does not agree with those of reference [23] claiming that the method built on the mode expansion does not adequately describe the critical behaviour of the finite system.

Note that by evaluating the Binder's cumulant ratio, we obtain a result, which have the same behaviour as a function of ε as that of the result of references [16,17]. It seems that to the first loop order, the method used here does not ameliorate the result obtained by means of the ε-expansion. It is possible that higher loop order can improve the result in comparison to the Monte Carlo method of reference [16].

In this paper, we concentrated our attention to a field theoretical model in the continuum (scaling) limit. Consequently the cutoff is send to infinity. By accounting the finite cutoff effects we expect that the finite-size scaling will be violated in a similar way to the case short-range interaction [22]. Note that the same effects can be obtained if one takes the model (1.1) with the parameter σ controlling the range of the interaction to be larger than the value, $\sigma = 2$, characterizing the short potential. This would verify whether the results obtained in references [32] at the spherical limit remains true for finite n.

Another possible extension of the results obtained here, in the static limit, is the application of the finite-size scaling theory to systems including critical dynamics. This can have direct implication to so systems exhibiting quantum critical behaviour.

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APPENDIX A: FINITE-SIZE CORRECTIONS TO THE BULK SUSCEPTIBILITY

In the appendix, we evaluate the finite-size correction to the bulk susceptibility in the region $L/\xi \gg 1$ for arbitrary dimension of the system. Here we use the mode expansion and we will work in the one loop order as explained in Section III. We are interested in particular in dimensions d, such that $\sigma < d < 2\sigma$. In the leading order of the non-zero ($\boldsymbol{k} \neq \mathbf{0}$) modes the effective Hamiltonian (3.13) reads

$$\mathcal{H}_{\mathrm{eff}}(r_0, u_0, L, \Phi^2) = \frac{1}{2} L^d [R\Phi^2 + \frac{1}{2} U \Phi^4 + \mathcal{O}(\Phi^6)], \tag{A1}$$

where

$$R = r_0 - r_{0c} + (n+2)u_0 \left[L^{-d} {\sum_{\boldsymbol{k}}}' (r_0 - r_{0c} + \boldsymbol{k}^\sigma)^{-1} - \int_{\boldsymbol{k}} \boldsymbol{k}^{-\sigma} \right], \tag{A2a}$$

$$U = u_0 - (n+8)u_0^2 L^{-d} {\sum_{\boldsymbol{k}}}' (r_0 - r_{0c} + \boldsymbol{k}^\sigma)^{-2}. \tag{A2b}$$

We have incorporated here the finite shift $r_{0c} = -(n+2)u_0 \int_{\boldsymbol{k}} \boldsymbol{k}^{-\sigma} + \mathcal{O}(u_0)$ of the parameter r_0 at one loop-order. Let us recall that in general one does a double expansion: one in the modes and the other in the coupling constant u_0. In other words the fact that we are working in fixed space dimension does not mean that the parameter u_0 should be kept fixed or we are not allowed to expand with respect to the small parameter u_0. This observation will be of a great importance in the following.

In the one-loop order the difference $r_0 - r_{0c}$ is proportional to the reduced temperature t i.e. $r_0 - r_{0c} = at$. In the following we will choose the coefficient $a = 1$. In terms of t, equations (A2) reads

$$R = t - (n+2)u_0 \Delta_1(t), \tag{A3a}$$

$$U = u_0 - (n+8)u_0^2 \int_{\boldsymbol{k}} (t + \boldsymbol{k}^\sigma)^{-2} + (n+8)u_0^2 \Delta_2(t), \tag{A3b}$$

where

$$\Delta_m = \int_{\boldsymbol{k}} (t + \boldsymbol{k}^\sigma)^{-m} - L^{-d} {\sum_{\boldsymbol{k}}}' (t + \boldsymbol{k}^\sigma)^{-m}. \tag{A4}$$

Now we will evaluate the susceptibility for the system confined to a finite geometry. In the present approximation it is

$$\chi = \frac{1}{n} \sqrt{\frac{L^d}{U}} \mathcal{G}_\chi \left(R \sqrt{\frac{L^d}{U}} \right), \tag{A5}$$

with

$$\mathcal{G}_\chi(z) = \frac{\int_0^\infty dx x^{n+1} \exp\left(-\frac{1}{2} z x^2 - \frac{1}{4} x^4\right)}{\int_0^\infty dx x^{n-1} \exp\left(-\frac{1}{2} z x^2 - \frac{1}{4} x^4\right)}. \tag{A6}$$

The integrals appearing in equation (A6) can be expressed in terms of the confluent hypergeometric function $U(a, b; z)$ according to

$$\int_0^\infty dy y^{\nu-1} e^{-z\frac{y^2}{2} - \frac{y^4}{4}} = \frac{1}{2} \Gamma\left(\frac{\nu}{2}\right) U\left(\frac{\nu}{4}, \frac{1}{2}; \frac{z^2}{4}\right), \qquad \text{for } z \geq 0. \tag{A7}$$

This function has a well known analytic properties (see reference [33]).

In the region $tL^{1/\nu} \gg 1$, corresponding to $z \gg 1$, using the asymptotic form of the function $\mathcal{G}(z)$ for large argument, which follows from that of the confluent Hypergeometric function, we obtain for the susceptibility

$$\chi = \frac{1}{R} \left[1 - (n+2) \frac{U}{L^d R^2} \right]. \tag{A8}$$

Here we would like to mention that this result is not the final expression for the susceptibility of the finite system. Here it easy to make a mistake by taking, to the lowest mode approximation, the coupling constants R and U to be equal to the initial coupling constants $r_0 - r_{0c}$ and u_0, respectively. One has to keep in mind that apart from the lowest mode approximation, a loop expansion in u_0 is involved in the calculations. So, here one must take into account the full expressions of the constants R and U up to the lowest (one-loop) order in u_0. Consequently by expanding in equation (A8) to order $\mathcal{O}(u_0^2)$ we arrive to the final result for the susceptibility

$$\chi = t^{-1} \left[1 - (n+2) u_0 \frac{L^{2\sigma-d}}{y^2} \left(1 + y F_{d,\sigma}(y)\right) \right]. \tag{A9}$$

Here we have used $y = tL^\sigma$ and $\Delta_1(t) \equiv -L^{2\sigma-d} F_{d,\sigma}(tL^\sigma)$. As one sees equations (3.34) and (A9) are similar and all what it has been written after equation (3.34) remains true here. The conclusion we can draw from here is that both *the ordinary perturbation theory* and the method based on *the mode expansion* give *only one and the same* result for the susceptibility in the region, where the bulk properties of the system are dominating the critical behaviour of the system i.e. in the region $L/\xi \gg 1$.

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